**A Mathematical Model of the Velocity of a Falling Water Drop in Air**

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## Abstract

Due to the fluidity of water, the kinematics and dynamics of falling water drops in air may not be as simple as those of common solid objects. A mathematical model for the velocity of a falling water drop with respect to time was proposed and refined using experimental data, and was determined to be representative of an actual water drop. This model was then compared to the behaviour of a water drop as predicted by classical Newtonian physics, and was found to be similar, suggesting that fluidity is not a large factor in the kinematics and dynamics of a falling water drop.

## Introduction

Water is an essential part of our lives, and falling water drops in air are a common phenomenon, particularly as rain. However, the kinematics and dynamics of falling water drops are not necessarily as simple as those of the solid objects often analyzed with classical Newtonian physics, due to the fluidity of water. Water may circulate internally in a water drop, and the shape of an entire water drop may change in response to external forces, such as drag force due to the air [1].

By proposing a mathematical expression for the velocity of a falling water drop with respect to time derived from basic Newtonian physics and refining it by fitting it to experimental data, we can create a model. Furthermore, we can determine whether the fluidity of water is a large factor in the motion of a water drop and if it can be taken into account in the model. As well, we can determine how representative the final model is of an actual water drop. In this experiment, the time required for a water drop to fall a certain distance was measured, requiring an expression for distance to be derived from the proposed expression for velocity so that the experimental data and mathematical model could be fitted and compared easily.

## Methods

The amount of time required for a water drop to fall from a certain height was determined for heights from 0.30 m to 6.30 m, at 0.30 m increments. At each increment, height was first measured, and then thirty measurements of time were taken.

*Measurement of height*

Heights less than 1.00 m were measured using a metre stick. Those greater than 1.00 m were measured using a homemade plumb-line, consisting of a long, thin string tied to a weight. The plumb-line was hung down and then measured with a metre stick. Heights less than 4.00 m were measured indoors, on a staircase, while greater heights were measured outdoors due to space limitations. All heights were measured near a wall so that a mark could be placed on the wall, ensuring that all measurements of time were taken at the same height.

*Measurement of time*
A plastic bucket with a flat bottom and vertical sides was placed on the ground, so that drops could be easily heard when they hit the bottom. A microphone, attached to a computer, was placed next to the bucket so that it could detect the sound. Measurements were taken using Audacity, a recording program on the computer, and a dropper positioned at the measured height. Tap water at room temperature was used. The dropper was squeezed slowly to minimize the initial velocity of water drops. When a drop became detached from the dropper, a recording was manually started. After it had fallen into the bucket, the recording was manually stopped. The resulting waveform was used to visually determine the length of time required for the drop to fall; the drop hitting the bottom of the bucket produced a sequence of spikes, as shown by Figure 1.

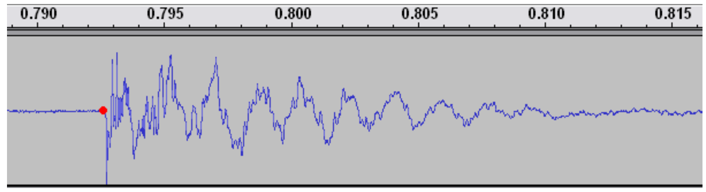

**Figure 1.** Sample waveform, produced by Audacity. Time required for a water drop to fall was measured at the base of the first spike, as shown by the red dot.

In order to obtain a measurement easily comparable to a mathematical model, average reaction time was also measured and added to each original measurement of time. Simple reaction time, the time required to respond to a stimulus, such as a water drop beginning to fall, was measured using an online tool [2]. Thirty measurements were taken and the average was calculated.

However, as height decreased, reaction time became a larger impediment to measurements. At 0.30 m, it was impossible to use a microphone to determine time, as the time required for the water drop to fall was less than the average reaction time. Thirty measurements of time were taken using a video camera, by recording a video of a drop falling and then playing the video slowly in order to determine time. A bright light was used to help visualize drops against a contrasting background. Despite this, the video was neither very clear nor smooth for small time intervals, resulting in a relatively large uncertainty in measurements at this height.

*Measurement of volume*
In order to compare the data to a mathematical model related to the mass of a water drop, measurement of mass was also required. As a water drop has a small mass, this was done by measuring the volume of three sets of fifty drops in a small graduated cylinder, and then converting the average volume of one drop to mass, using the literature value of the density of water, 0.9970 g/cm$^3$ at standard conditions [3].

## Results and Discussion
Height was plotted against the average time required for water drops to fall at each height, which was added to my average measured simple reaction time of 0.216 s, as shown in Figure 2, below. As expected, the resulting plot was curved for small times, due to acceleration of the drops, and generally linear for larger times, after terminal velocity had been reached.

A mathematical model was constructed by assuming two forces acting on a water drop: the downward force of gravity and an upward drag force, related to velocity and a constant, which will be referred to as the drag coefficient, $\gamma$. An expression for acceleration, $a(t)$, was determined, as shown in Equation 1, so that acceleration downwards decreases with time $t$, until terminal velocity is reached with $a(t)=0$. This was based on classical Newtonian physics, which states that a solid, falling object will accelerate downwards due to the force of gravity, with acceleration downwards decreasing with time due to the upwards drag force, until terminal velocity is reached.

$$a(t) = \frac{dv}{dt} = \frac{\gamma}{m}v^2 - g, \qquad (1)$$

where $v$ is the velocity of a water drop and $g$ is acceleration due to gravity.

Integration was used to derive an expression for velocity, $v(t)$, from Equation 1, assuming that $v(0)=0$ m/s, as shown by Equation 2. Although the velocity of each water drop as it became detached from the dropper was likely not exactly 0 m/s, due to the dropper not being held perfectly steady as well as slight applied air pressure inside the dropper, it was assumed that the difference is negligible.

$$v(t) = \frac{dx}{dt} = \sqrt{\frac{mg}{\gamma}}\,\frac{e^{2t\sqrt{\frac{\gamma g}{m}}} - 1}{e^{2t\sqrt{\frac{\gamma g}{m}}} + 1}. \qquad (2)$$

As expected, this equation also shows that a water drop accelerates until it reaches terminal velocity, for all values of the mass and drag coefficient. By determining the limit of $v(t)$ as $t$ approaches infinity, it is also clear that a smaller mass results in a lower terminal velocity.

In order to make this mathematical model comparable to experimental data, integration was used again to derive an expression for height, or vertical distance, $x(t)$, from Equation 2, assuming that $x(0)=0$ m, as shown by Equation 3. The distance from the measured height of each drop as they became detached from the dropper was also likely not exactly 0 m, due to the dropper not being held perfectly steady. This difference is taken into account in the uncertainty in height.

$$x(t) = \frac{m}{2\gamma}\left[\ln\left(e^{2t\sqrt{\frac{\gamma g}{m}}} + 1\right) - \ln\left(\frac{e^{2t\sqrt{\frac{\gamma g}{m}}}}{e^{2t\sqrt{\frac{\gamma g}{m}}} + 1}\right) - \ln 4\right]. \qquad (3)$$

The mass of an average water drop, $m$, was determined to be $8.33\times10^{-3}$ kg, as shown in Table 1. This is comparable to the size of a large rain drop [4].

| | |
|---|---|
| Average volume of 50 drops (mL) | 4.13 ± 0.05 |
| Average volume of 1 drop (mL) | 0.0826 ± 0.001 |
| Average mass of 1 drop, $m$ (kg) | $8.33\times10^{-3} \pm 1\times10^{-6}$ |
| Drag coefficient, $\gamma$ (kg/m) | $7.0\times10^{-6}$ |

**Table 1.** Constants in the mathematical model.

The experimental data was fitted to the mathematical model provided by Equation 3 by changing the value of the drag coefficient in order to minimize the value of $\chi^2$. The formula for $\chi^2$ is given by Equation 4, where $x_i$ is the measured value of height and $x(t)_i$ is the value of height as determined by the model at time $t$, and $N$ is the total number of measurements of height.

$$\chi^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - x(t)_i)^2. \qquad (4)$$

The drag coefficient was determined to be $7.0\times10^{-6}$ kg/m, as shown in Table 1, above.

These constants were used to determine a refined model based on Equation 3, which was plotted with height versus the average time required for water drops to fall, as shown in Figure 2.

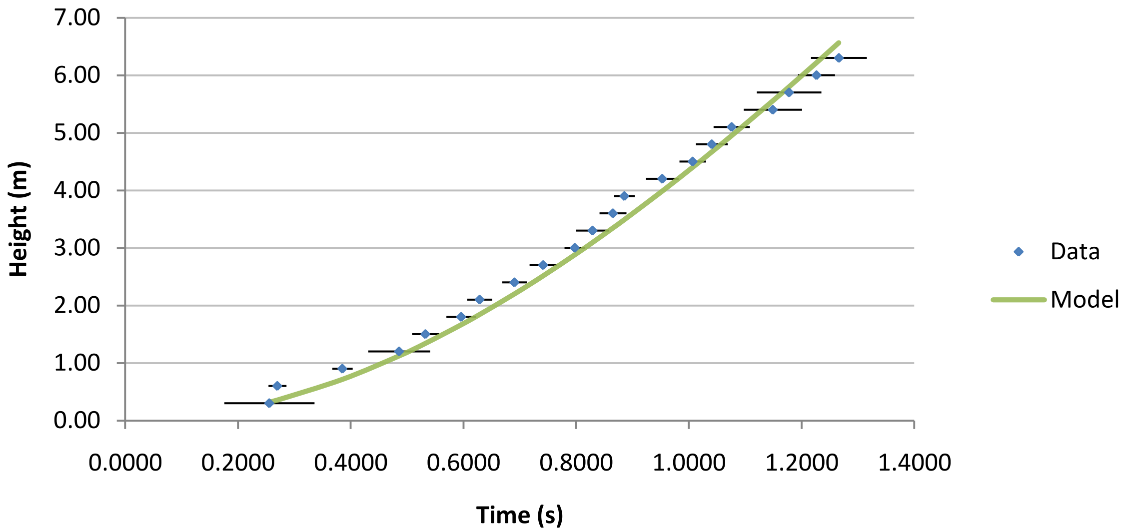


**Figure 2.** Data points are the average time (s) required for water drops to fall at each height (m), added to my average simple reaction time of 0.216 s. Vertical error bars represent uncertainty in height. Most are not visible as they are within the data points. Horizontal error bars represent uncertainty in time, as determined by standard deviation. The model is shown by the curve.

Uncertainty in height is primarily due to measuring with both the metre stick and plumb-line. The metre stick was held as vertically as possible while measuring height, but was likely not perfectly vertical, resulting in errors in measurement. Errors may have resulted due to the string of the plumb-line being pulled with a slightly different force by the weight when hanging down as compared to by hand when being measured with the metre stick. Furthermore, when height was measured outdoors, the wind may have resulted in the plumb-line not hanging perfectly vertically and may have blown water drops horizontally. However, since only drops falling into the bucket were measured, drops that were blown significantly horizontally have no effect on the data. Uncertainty is also due to the dropper not being held perfectly steady, which may have resulted in slight shifts vertically.

Uncertainty in time is due to measurements with the microphone and video camera, as well as measurement of simple reaction time. The error resulting from determining time from the waveform is small, due to the high degree of precision provided by the software. The error associated with the starting of the recording is presumably due to reaction time, and is therefore taken into account in the reaction time uncertainty, as simple reaction time was added to time measured with the microphone. However, standard deviation in time measured with the microphone was generally greater than reaction time uncertainty, likely due to the difficulty in determining when the water drop became detached from the dropper. Because of this, standard deviation is used as uncertainty in time measured with the microphone. Uncertainty in time measured with the video camera was greater, due to the video being neither very clear nor smooth for small time intervals.

All data points except those at 0.60 ± 0.01 m, 0.90 ± 0.01 m, 2.10 ± 0.01 m, 2.40 ± 0.01 m, 3.60 ± 0.03 m, and 3.90 ± 0.03 m, agree with the mathematical model, with uncertainty taken into account. The discrepancies at 0.60 ± 0.01 m and 0.90 ± 0.01 m are likely due to reaction time, as reaction time was relatively large compared to the times measured at these heights. The discrepancies at greater heights are likely due to positions from which measurements were taken, on a staircase, which resulted in it being slightly more difficult to control the dropper and to see the drops detaching from the dropper.

Overall, this general agreement between the experimental data and the refined model for height, provided by Equation 3, supports the mathematical model proposed for the velocity of a water drop with respect to time, Equation 2, for the determined values of the mass of a water drop and drag coefficient. Furthermore, it provides support for the model as a realistic representation of an actual water drop for the determined values of the constants, as well as for a range of masses, if drag coefficients are determined by fitting data to the model. As well, Equation 1, the expression for the acceleration of a water drop with respect to time, from which Equations 2 and 3, the expressions for the velocity and height, were derived, was based on classical Newtonian physics and the assumption that the falling object is solid. Because of this, the agreement between the experimental data and the model also suggests that the fluidity of a water drop, resulting in the ability to circulate water internally and to change shape in response to external forces, is not a large factor in its kinematics and dynamics. Discrepancies between the motion predicted by classical Newtonian physics and the actual motion of a water drop falling in the air, likely due to the fluidity of water, may be taken into account with the drag coefficient.

## Conclusions

The mathematical model proposed for the velocity of a falling water drop in air was refined and found to be representative of an actual water drop, particularly for the determined values of the mass of a water drop and drag coefficient. This suggests that the internal circulation of water within a drop and the ability of an entire drop to change shape in response to external forces are not large factors in the kinematics and dynamics of a water drop, and that their effects may be taken into account with the drag coefficient. It also shows that mathematical expressions based on classical Newtonian physics, fitted to experimental data by changing the value of drag coefficient, may be sufficient in describing the acceleration, velocity, and vertical height of a falling water drop in air.

## References

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